A Systems Approach to Biology

MCB 195

Lecture 3
Tuesday, 10 Feb 2005

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DYNAMICAL SYSTEMS

steady states, stability

LINEAR SYSTEMS

$$\frac{dx}{dt} = \lambda - dx - \beta xy + bn(n-1)y$$

$$\frac{dy}{dt} = -ay + b(z+y) - 2nby$$

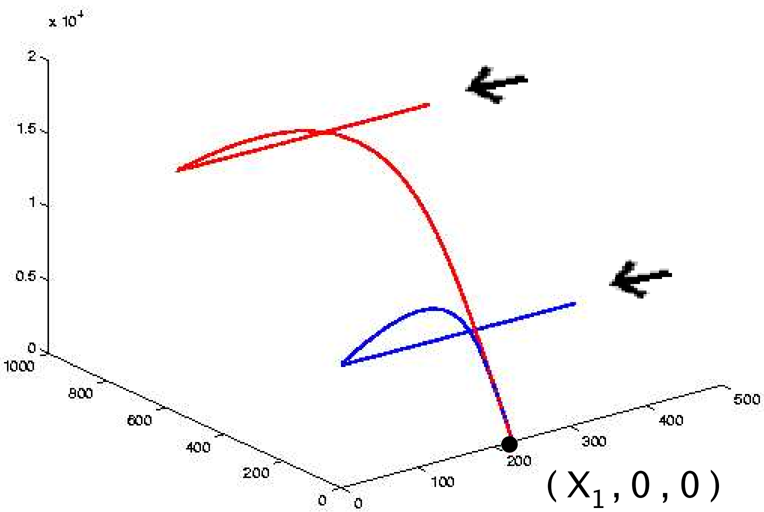
$$\frac{dz}{dt} = \beta xy - az - n(n-1)by$$

$X_1 = \lambda/d$

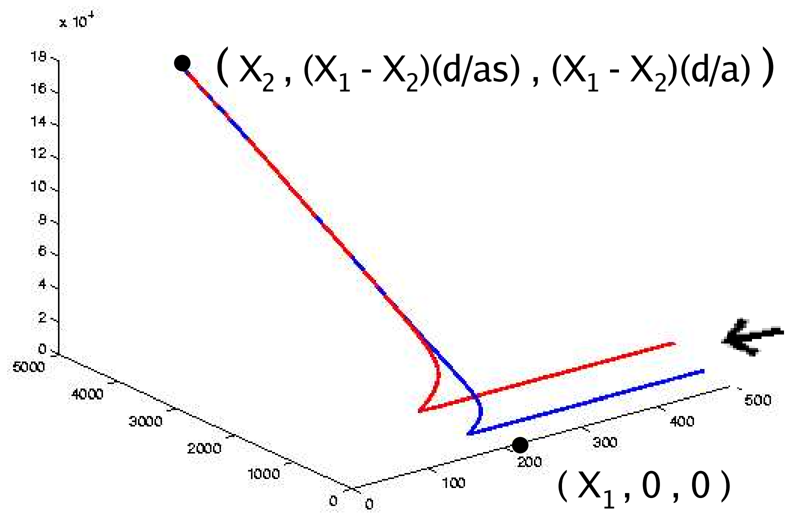
$X_2 = \dfrac{(a + (n-1)b)(a + nb)}{b\beta}$

$s = a/b + 2n-1$

$X_1 \leq X_2$



$X_1 > X_2$



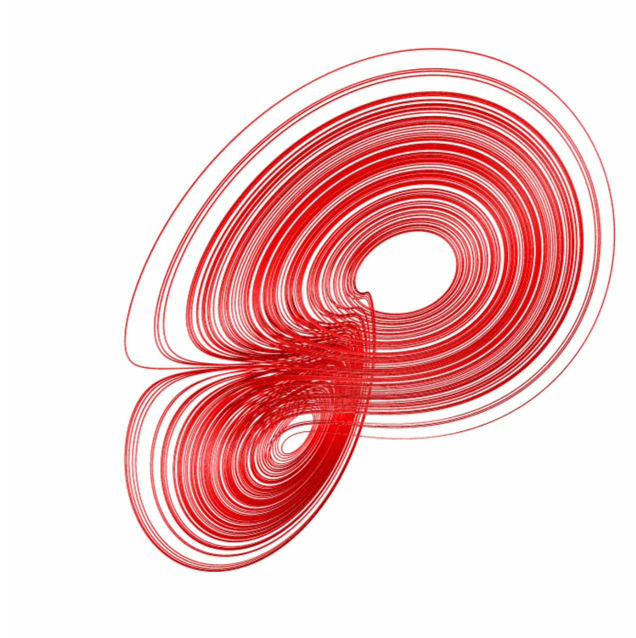
changes in processes "outside" the system, like clearance or degradation, can have profound consequences on system behaviour through BIFURCATIONS

Nonlinear dynamical systems can be so complicated that we have given up on the search for explicit solutions

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = -rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

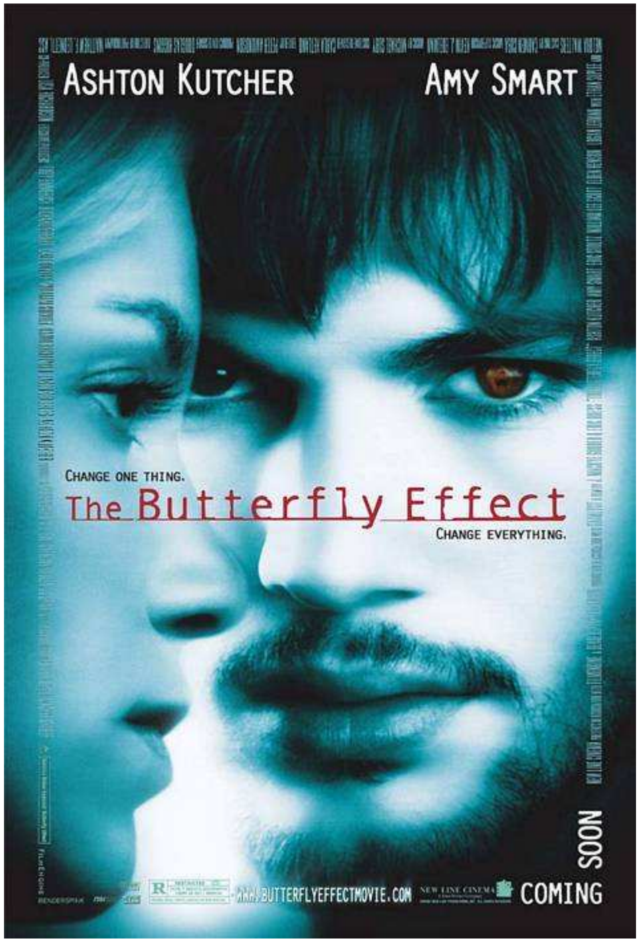


CHAOS

sensitive dependence on initial conditions

Ed Lorentz (1972):

"Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"



THE DYNAMICAL SYSTEMS PERSPECTIVE

comes to biology from mathematics and physics

explict description of state

description of how the state evolves in time

discrete
state variables are Boolean
evolution described by logical functions

stochastic
state variables are probability distributions
evolution described by stochastic DEs

spatial
state variables are functions of space and time
evolution described by PDEs

continuous
state variables are functions of time
evolution described by ODEs

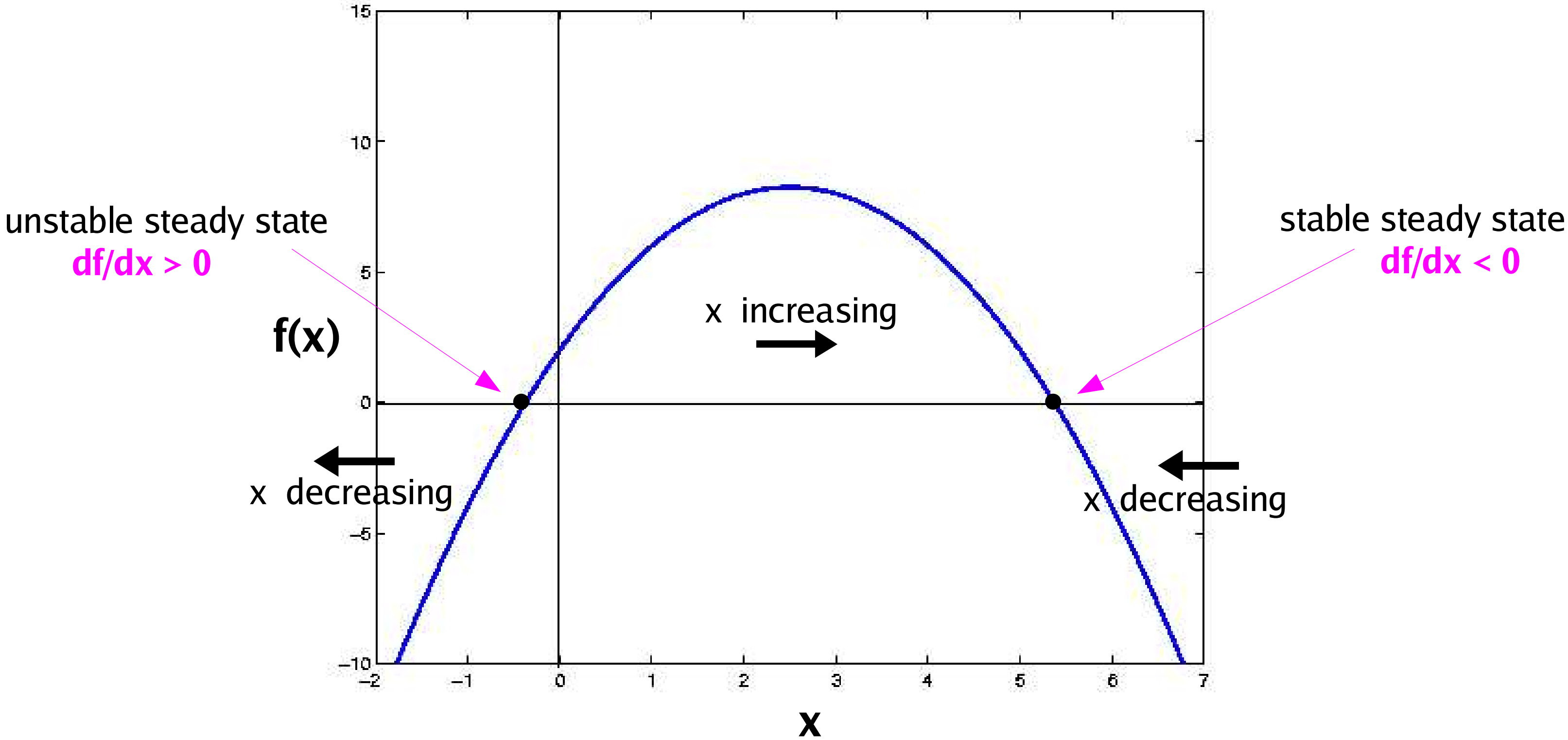
this is not the only perspective on systems

in control theory, signal processing, etc systems are “input-output” systems

How do we work out the behaviour of a dynamical system?

3. determine the local **stability** of the steady states

1 dimensional dynamical system $\frac{dx}{dt} = f(x)$



stable — any sufficiently small perturbation from the steady state always returns

unstable — not stable

1 dimensional dynamical system $\dfrac{dx}{dt} = f(x)$

1. find a steady state $x = x_{st}$ $\left(\dfrac{dx}{dt}\right)_{x = x_{st}} = 0$

2. calculate $\left(\dfrac{df}{dx}\right)_{x = x_{st}}$

3. if negative then x_{st} is stable

4. if positive then x_{st} is unstable

5. **BUT** if zero then x_{st} could be stable or unstable

1 dimensional dynamical systems cannot oscillate

n dimensional dynamical system $\frac{dx}{dt} = f(x)$

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \dots , x_n)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2, \dots , x_n)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(x_1, x_2, \dots , x_n)$$

$$f = \mathbb{R}^n \text{ ----> } \mathbb{R}^n$$

$$Df = \left(\frac{\partial f_i}{\partial x_j}\right)$$

derivative of f
"Jacobian"

n x n matrix
of partial derivatives

n dimensional dynamical system $\dfrac{dx}{dt} = f(x)$

1. find a steady state $x = x_{st}$ $\left(\dfrac{dx}{dt}\right)_{x = x_{st}} = 0$ so that $f(x_{st}) = 0$

2. calculate the Jacobian matrix at the steady state $A = (Df)_{x = x_{st}}$

3. if all the eigenvalues of A have negative real part then x_{st} is stable

4. if x_{st} is hyperbolic and at least one of the eigenvalues of A has positive real part then x_{st} is unstable

none of the eigenvalues of A has zero real part

$$\frac{dx}{dt} = \lambda - dx - \beta xy + bn(n-1)y$$

$$\frac{dy}{dt} = -ay + b(z+y) - 2nby$$

$$\frac{dz}{dt} = \beta xy - az - n(n-1)by$$

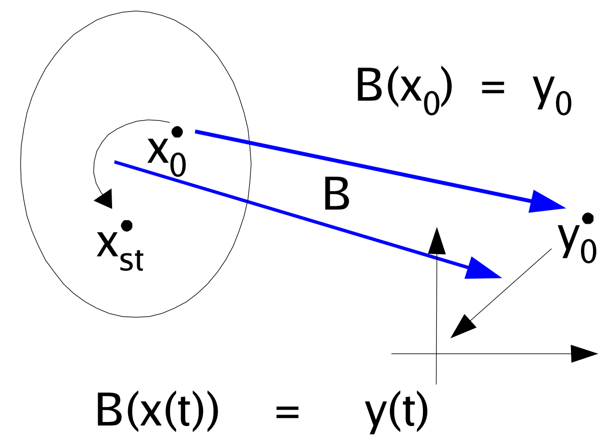
$$\begin{pmatrix} -d - \beta y & -\beta x + n(n-1)b & 0 \\ 0 & -a - (2n-1)b & b \\ \beta y & \beta x - n(n-1)b & -a \end{pmatrix}$$

$$\frac{dx}{dt} = f(x)$$ nonlinear system

$$f(x_{st}) = 0$$ steady state

$$A = (Df)_{x = x_{st}}$$ Jacobian matrix

$$\frac{dy}{dt} = Ay$$ linearised system



Hartman-Grobman Theorem

the behaviour of the trajectories close to a hyperbolic steady state
are qualitatively similar to those of the linearised system

LINEAR DYNAMICAL SYSTEMS

$$\frac{dx}{dt} = ax$$ 1 dimensional scalar equation

solution is $x(t) = \exp(at)x_0$ $\exp(u) = 1 + u + u^2/2! + u^3/3! + \ldots$

$$\frac{dx}{dt} = Ax$$ n dimensional matrix equation

solution is $x(t) = \exp(At)x_0$ $\exp(U) = I + U + U^2/2! + U^3/3! + \ldots$

identity matrix

matrix product !!!

exp(U) is the matrix exponential $\qquad$ *expm*(U) in MATLAB

for scalars a, b $\qquad \exp(a + b) = \exp(a)\exp(b)$

for matrices A, B $\qquad \boxed{\exp(A + B) = \exp(A)\exp(B) \quad \textbf{ONLY IF } AB = BA}$

The behaviour of the linear dynamical system $\dfrac{dx}{dt} = Ax$

is qualitatively similar to that of $\dfrac{dy}{dt} = BAB^{-1}y$

because

$$\boxed{\exp(BAB^{-1}) = B\exp(A)B^{-1}}$$

if $y(0) = Bx(0)$ then $y(t) = Bx(t)$

A and BAB^{-1} have the same eigenvalues

There are only 3 possibilities for a 2 x 2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

A	BAB^{-1}	$\exp(BAB^{-1})$
2 complex conjugate eigenvalues	$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$	$\exp(a)\begin{pmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{pmatrix}$
2 distinct real eigenvalues	$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$	$\begin{pmatrix} \exp(a_1) & 0 \\ 0 & \exp(a_2) \end{pmatrix}$
2 equal real eigenvalues	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$	$\exp(a)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad Tr(A) = a + d \qquad det(A) = ad\text{-}bc \qquad \Delta = Tr(A) - 4det(A)^2$$