A systems approach to biology

SB200

Lecture 4

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Recap of Lecture 3

STABILITY TH.
LINEARISATION TH.

Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

matrix algebra

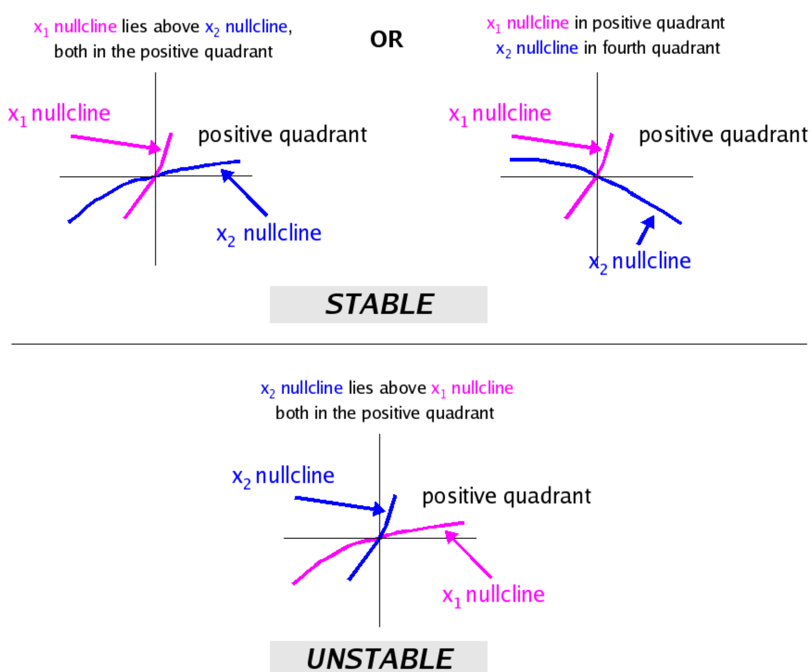
A^{-1} exists, iff, $\det A \neq 0$

eigenvalues

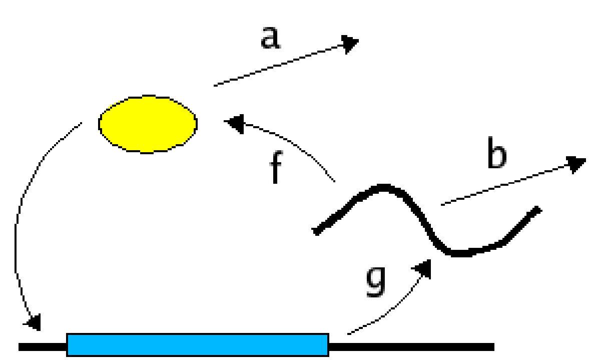
$Ax = ux$

$\det(A - uI) = 0$

$\det AB = (\det A).(\det B)$



stability of an autoregulatory loop



$$\boxed{\frac{dx}{dt} = f(x) \quad \text{with steady state at } x = x_{st}}$$

1 dimensional

$$a = \left(\frac{df}{dx}\right)\bigg|_{x=x_{st}}$$

$dx/dt = ax$

$$\boxed{x(t) = \exp(at)x_0}$$

$\exp(a) = 1 + a + a^2/2 + a^3/3! + \ldots$

exponential

$\exp(a+b) = \exp(a)\exp(b)$

n > 1 dimensional

$$A = (Df)|_{x=x_{st}}$$

$dx/dt = Ax$

$$\boxed{x(t) = \exp(At)x_0}$$

$\exp(A) = I + A + A^2/2 + A^3/3! + \ldots$

matrix exponential

$\exp(A+B) = \exp(A).\exp(B)$

provided $AB = BA$

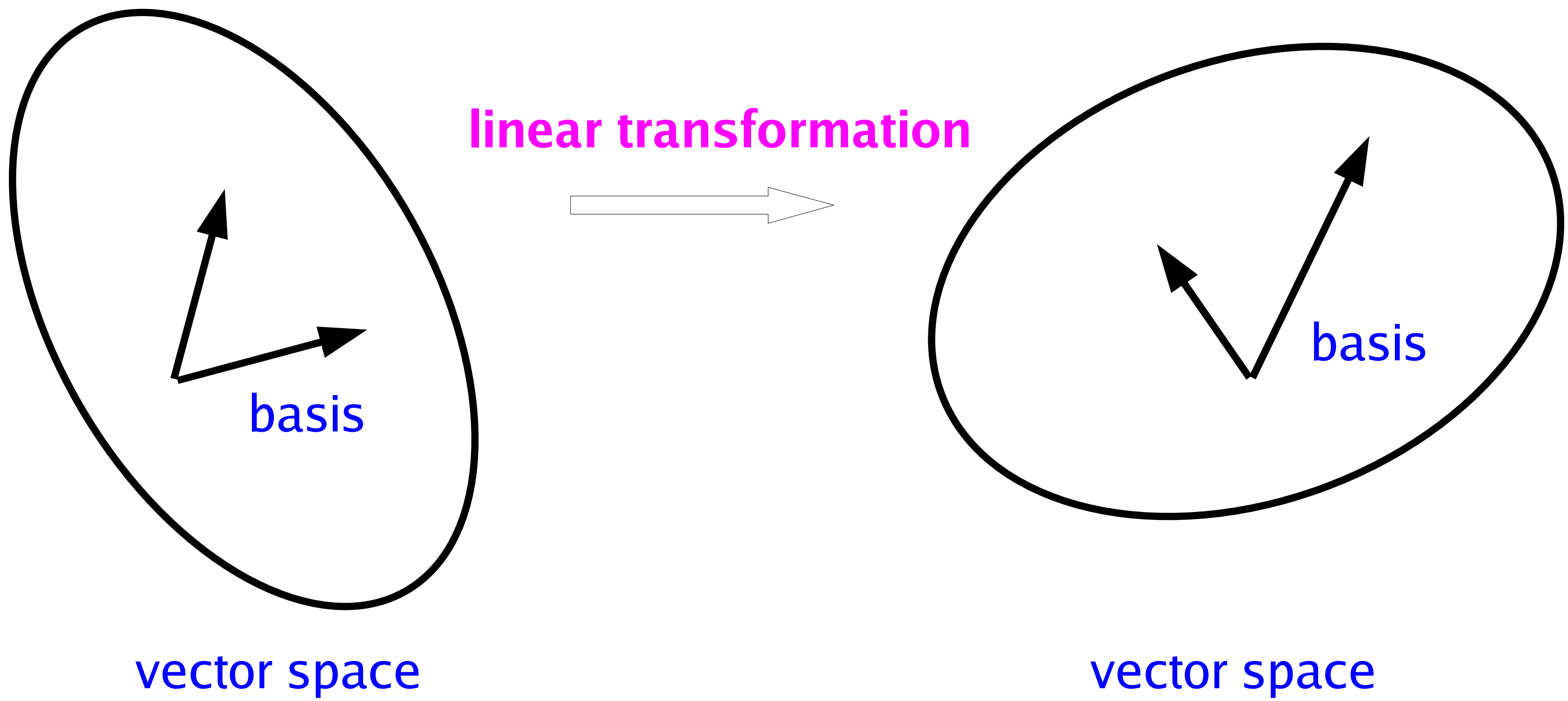
calculating exp(A)

matrices are like numbers –
they can be added and multiplied together

except that multiplication is not commutative
and not all non-zero matrices have an inverse

*matrices ALSO represent **linear transformations***
*with respect to a chosen coordinate **basis***

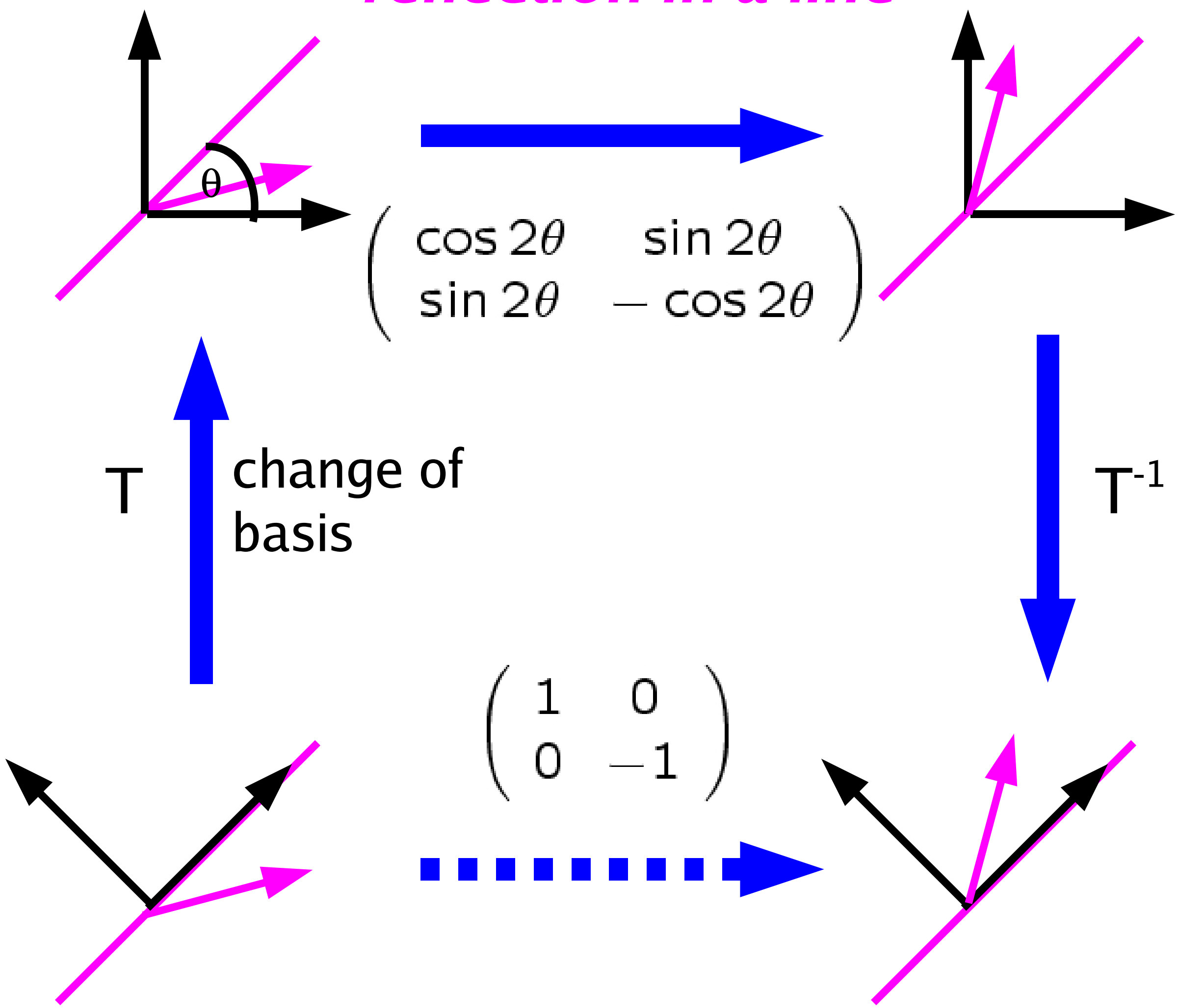
eigenvectors (when they exist) can give a basis
in which the matrix is particularly simple



$$\mathbf{L}(\mathbf{x} + \mathbf{y}) = \mathbf{L}(\mathbf{x}) + \mathbf{L}(\mathbf{y})$$

$$\mathbf{L}(\lambda\mathbf{x}) = \lambda\mathbf{L}(\mathbf{x})$$

reflection in a line



Change of basis corresponds to similarity of matrices

A and B are **SIMILAR** if $B = T^{-1}AT$

If A and B are similar then A and B have the same

characteristic polynomial
eigenvalues
det and **Tr**

How do we calculate exp(A)?

Find a change of basis in which A becomes simple

$$\exp(T^{-1}AT) = T^{-1}\exp(A)T$$

A = any 2 x 2 matrix

Δ = discriminant of the characteristic polynomial of A

The simple case $\Delta > 0$

eigenvalues are real and distinct

$$\xrightarrow{\text{change of basis}} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

ie: reflection

The complex case $\Delta < 0$

eigenvalues are complex and conjugate

$$\xrightarrow{\text{change of basis}} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

ie: rotation

The awkward case $\Delta = 0$

eigenvalues are real and equal

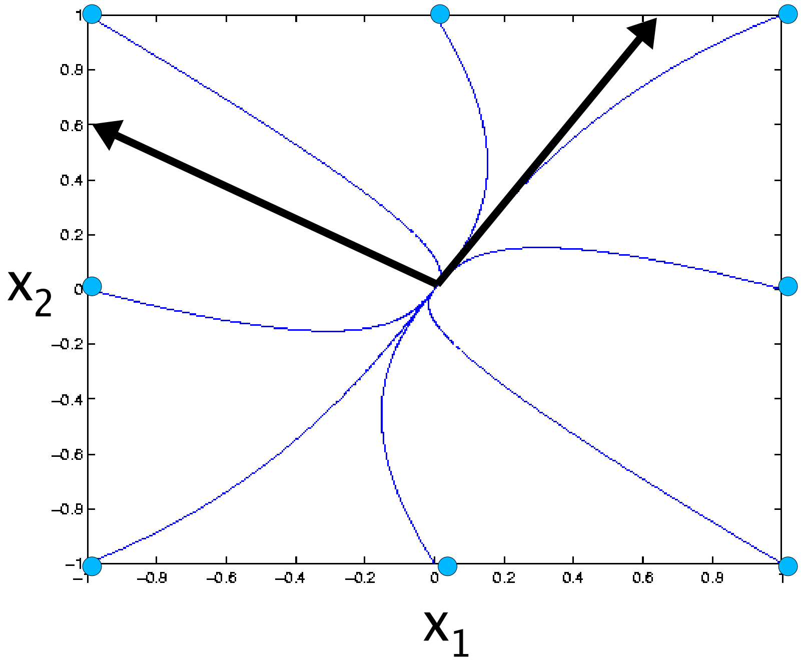
$$\xrightarrow{\text{change of basis}} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

simple case   disc > 0   both eigenvalues negative

$$\exp\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \exp a & 0 \\ 0 & \exp b \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}$$

Tr = -5 det = 5 disc = 5
eigenvalues = -3.62, -1.38

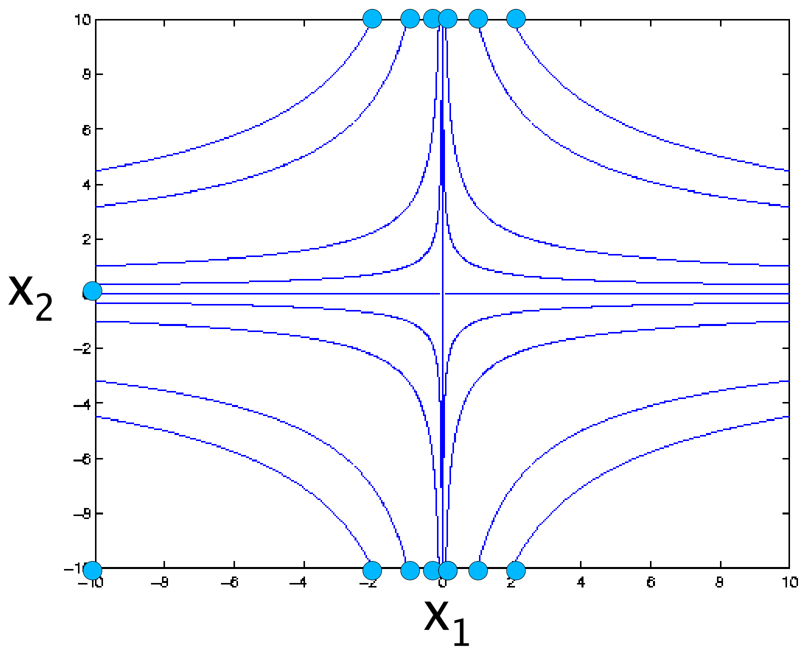


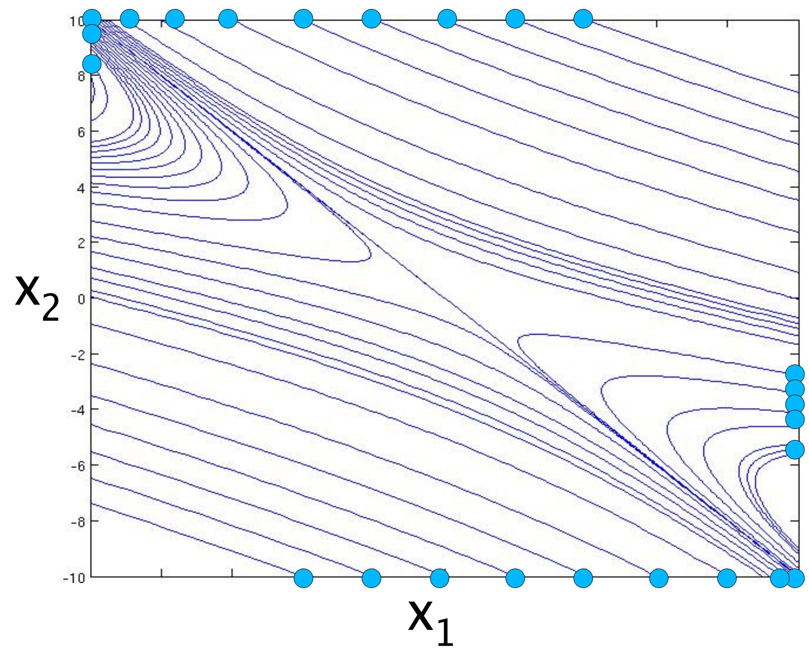
stable node

simple case   disc > 0   one eigenvalue positive

$$\exp\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \exp a & 0 \\ 0 & \exp b \end{pmatrix}$$

saddle





$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \xleftarrow{\text{similarity}} \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$$

Tr = 1 **det = -2** disc = 9

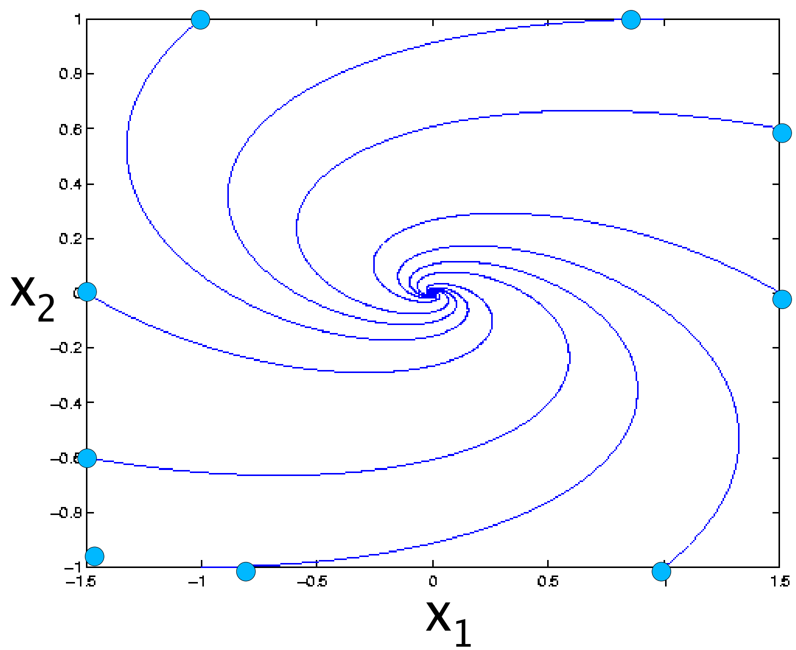
eigenvalues = 2, -1

complex case disc < 0 complex eigenvalues

$$\exp\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \exp(a)\begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$$

$$\begin{pmatrix} -2 & -5 \\ 1 & -1 \end{pmatrix}$$

Tr = -3 det = 7 **disc = -19**
eigenvalues = -1.5 ± 2.18 *i*



stable spiral

Complex eigenvalues imply (damped) oscillation, with frequency given by the imaginary part of the eigenvalue