A simple production-consumption system

This note is for SB200, "A systems approach to biology". It provides more details of the analysis of the production-consumption system that was done in the first lecture, on 19 September 2006. If you have any questions, write to me (jeremy@hms.harvard.edu) or speak to one of the TAs.

I explained in class how the production-consumption system leads to the differential equation

$$\frac{dx}{dt} = a - \delta x \; . \tag{1}$$

In the lecture and on the slides, I used d in place of δ, for the first-order consumption rate constant. As someone pointed out in class, this is not a very good notation because dx could mean either the differential dx or d times x. In this note, I have swapped d for δ to avoid any confusion. One other point to keep in mind is that the *state variable* x in (1) is really a function of time, t. I should write it $x(t)$ to indicate this explicitly but it gets awfully boring to do this all the time, so I usually do not, unless it is necessary to emphasise it. By the same token, a and δ are *parameters*, not state variable. These are constant (ie: just numbers) and are not functions of time.

Equation (1) looks very close to the standard differential equation for an exponential function

$$\frac{dx}{dt} = \delta x \; ,$$

whose solution is $x(t) = A\exp(\delta t)$. (**If this does not seem familiar, you really need to brush up your calculus! You should speak to Tom Kolokotrones or one of the other TA's about getting some help with this.**) Note that the constant A depends on the initial conditions. In particular, if we set $t = 0$, we find that $A = x(0)$. In other words, A is the initial amount of x at time 0.

The only difference between the two equations is the constant term a in (1). We can try and transform (1) using a change of variable to get rid of the a. It is easy to see that if we make the substitution

$$x = \frac{y + a}{\delta} \tag{2}$$

then (1) simplifies to give

$$\frac{dx}{dt} = a - \delta\left(\frac{y + a}{\delta}\right) = -y \; . \tag{3}$$

This equation still involves x on the left hand side so we need to replace this using y. If we differentiate (2) with respect to t, we see that

$$\frac{dx}{dt} = \frac{1}{\delta}\frac{dy}{dt} \; .$$

If we put this together with (3) we find that

$$\frac{dy}{dt} = -\delta y \; ,$$

which is now in exactly the form which we know how to solve. The solution is

$$y(t) = A\exp(-\delta t) \; . \tag{4}$$

We now have to use (2) in reverse, to express (4) in terms of x instead of y. The first thing to determine is A. We see from (4) that $A = y(0)$ and from (2) that $y(0) = \delta x(0) - a$. We also see by rewriting (2) that $y(t) = \delta x(t) - a$. Putting these all together in (4) we find that

$$\delta x(t) - a = (\delta x(0) - a)\exp(-\delta t) \; ,$$

which, if we simplify it, gives

$$x(t) = \frac{a}{\delta} + \left(x(0) - \frac{a}{\delta}\right)\exp(-\delta t) \; .$$

This is what we derived in class.