dynamic processes in cells (a systems approach to biology)

jeremy gunawardena
department of systems biology
harvard medical school

lecture 3

8 september 2016

solving linear ODEs of order 1

the first-order, linear ODE

$$\frac{dx}{dt} = ax$$

has the unique solution

initial condition ↓

$$x(t) = e^{at} x(0)$$

where the exponential function $e^t = \exp(t)$ is defined by

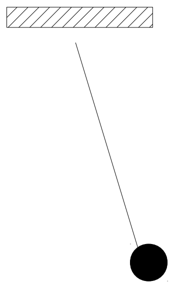
$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3.2} + \cdots + \frac{t^n}{n!} + \cdots$$

(this definition works generally for complex numbers or matrices *)

* see the lecture notes “Matrix algebra for beginners, Part III”

solving linear ODEs of order 2

pendulum



the second-order, linear ODE $$\frac{d^2x}{dt^2} = -ax \quad a > 0$$

initial conditions

has solutions $\cos(\sqrt{a}t)$ for x(0) = 1 and $(dx/dt)|_{t=0} = 0$

and $\sin(\sqrt{a}t)$ for x(0) = 0 and $(dx/dt)|_{t=0} = \sqrt{a}$

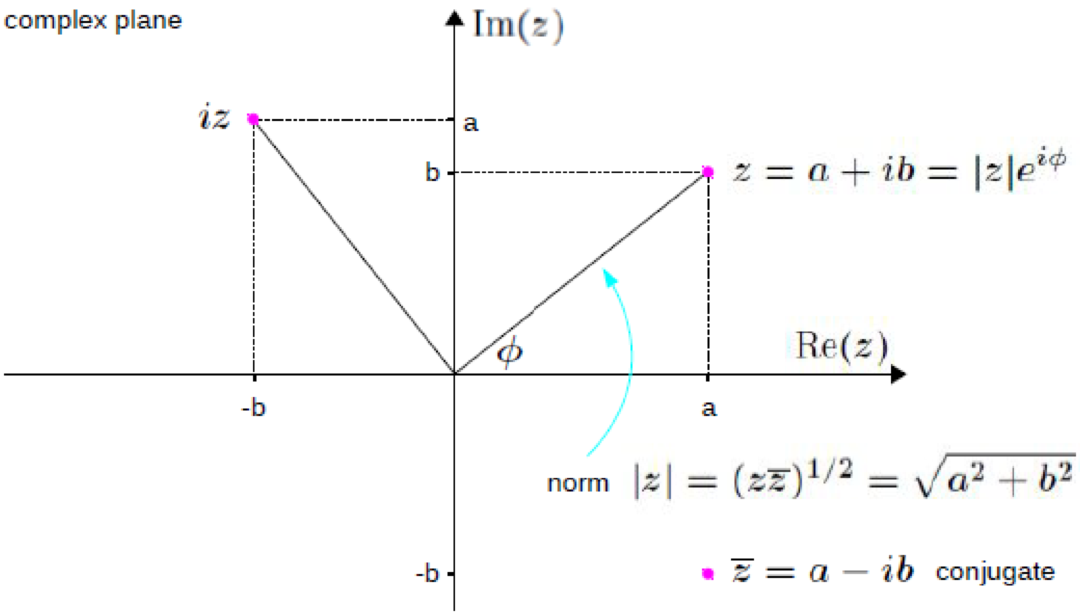
or, as power series,

$$\cos(t) = \sum_{k=0}^{\infty}(-1)^k \frac{t^{2k}}{(2k)!} \qquad \sin(t) = \sum_{k=0}^{\infty}(-1)^k \frac{t^{2k+1}}{(2k+1)!}$$

the complex numbers

numbers written in the form $a + ib$ with a, b ordinary “real” numbers and $i^2 = -1$

complex plane



Euler's formula

$$e^{it} = \cos(t) + i\sin(t)$$

1707-1783

the trigonometric functions are exponential functions over the complex numbers

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \qquad \sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

solving linear ODEs of order n

COMPLEX EXPONENTIALS ARE ALL YOU NEED (ALMOST*)

the solution of any linear ODE, no matter what its order, is (almost*) a linear combination of complex exponentials

$$\sum_i \lambda_i e^{z_1 t}$$

where z_i are complex numbers determined by the coefficients and λ_i are complex numbers determined by the initial conditions

* you also need powers of t $t^j e^{z_i t}$

the Laplace transform

1749-1827

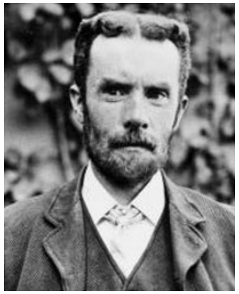
transforms a function $x(t)$ of t into a function $(\mathcal{L}x)(s)$ of s

$$(\mathcal{L}x)(s) = \int_0^\infty e^{-st} x(t) dt \quad Re(s) > c$$

the integral is defined ("converges") for **s** sufficiently large, provided $x(t)$ does not increase "too fast"

s may take complex values, in which case the integral is defined for the real part of **s** sufficiently large – so that the Laplace transform is defined in some right half plane of the complex numbers

using it to solve linear ODEs



1850-1925

the Laplace transform converts **differentiation by t** into **multiplication by s**, thereby transforming calculus into algebra

1. **apply the Laplace transform**

2. **express the Laplace transform of the solution as a function of s**

3. **break up this function of s into a linear combination of functions whose Laplace transforms are known (provided in a Table)**

4. **use the Table to write down the solution**

5. **remember Oliver Heavside**

Paul Nahin, **Oliver Heaviside. The Life, Work and Times of an Electrical Genius of the Victorian Age,** Johns Hopkins University Press, 1988; Bush, **Operational Circuit Analysis**, John Wiley, 1929

Vaneevar Bush, **Operational Circuit Analysis**, John Wiley, 1929. Bush wrote that readers would *"frequently turn for inspiration and background to Heaviside's own works, of which this is in some sense an interpretation"*

properties of the Laplace transform

1. **it converts differentiation by t into multiplication by s**

$$\mathcal{L}\left(\frac{df}{dt}\right) = s(\mathcal{L}f) - f(0)$$

initial condition

2. **it converts multiplication by t into differentiation by s**

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}(\mathcal{L}f)(s)$$

properties of the Laplace transform

3. it is linear

$$\mathcal{L}(\lambda_1 x_1(t) + \lambda_2 x_2(t)) = \lambda_1 (\mathcal{L}x_1)(s) + \lambda_2 (\mathcal{L}x_2)(s)$$

4. it is one-to-one (for our purposes)

if $(\mathcal{L}f)(s) = (\mathcal{L}g)(s)$ then $f(t) = g(t)$

$f(t)$	$(\mathcal{L}f)(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{d^n f}{dt^n}$	$s^n(\mathcal{L}f)(s) - \sum_{r=0}^{n-1} s^{n-r-1}\frac{d^r f}{dt^r}(0)$

TABLE of Laplace transforms

initial conditions

solving linear ODEs with the Laplace transform

$$\frac{dx}{dt} + ax = b$$

1. apply the Laplace transform to both sides

$$\mathcal{L}(\frac{dx}{dt} + ax) = \mathcal{L}(b)$$

2. use the properties of $\mathcal{L}$ and the TABLE to simplify and solve for $\mathcal{L}x$

$$s(\mathcal{L}x) + a(\mathcal{L}x) = \frac{b}{s} + x(0)$$

$$(\mathcal{L}x) = \frac{b}{s(s+a)} + \frac{x(0)}{s+a}$$

solving linear ODEs with the Laplace transform

$$(\mathcal{L}x) = \frac{b}{s(s+a)} + \frac{x(0)}{s+a}$$

3. use **partial fractions** to rewrite the RHS in terms of functions in the TABLE

$$\frac{b}{s(s+a)} = \left(\frac{b}{a}\right)\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

4. now use the TABLE again to deduce what the solution must have been

$$x(t) = \frac{b}{a} + (x(0) - \frac{b}{a})e^{-at}$$

partial fractions

3. use partial fractions to rewrite the RHS in terms of functions in the TABLE

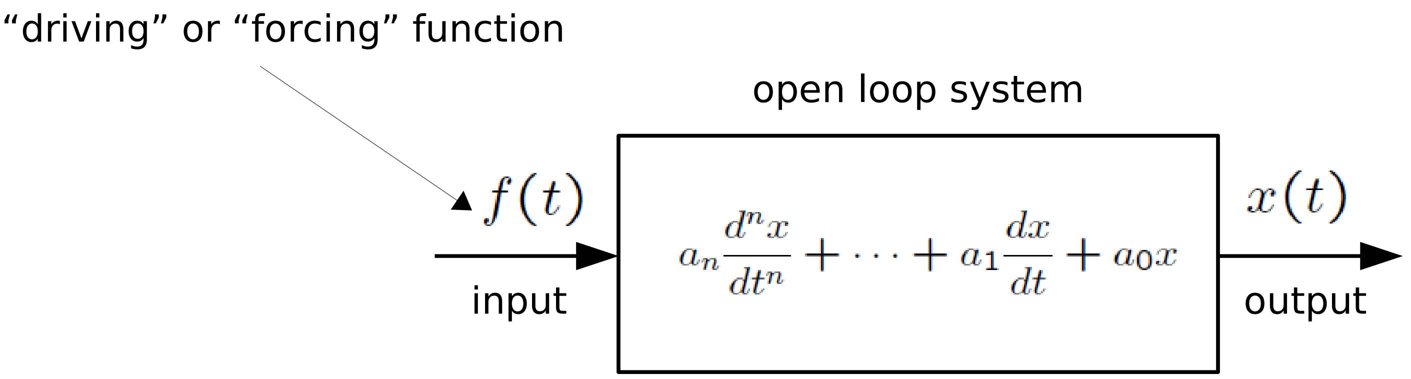
$$\frac{b}{s(s+a)} = \left(\frac{b}{a}\right)\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

this depends on the **linear factors** of the denominator polynomial

$$s^2 + as = s(s+a)$$

which determine where the denominator polynomial becomes 0, at s = 0 and s = -a

the general linear system



$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

applying the Laplace transform

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

the Laplace transform converts differentiation by ***t*** into multiplication by ***s***

$$\mathcal{L}\left(\frac{d^j x}{dt^j}\right) = s^j (\mathcal{L}x)(s) - \sum_{u=0}^{j-1} s^{j-u-1} \frac{d^u x}{dt^u}(0)$$

depends only on the initial conditions

applying the Laplace transform to both sides of the equation

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0)(\mathcal{L}x) - c(s) = (\mathcal{L}f)(s)$$

resembles the original differential equation

depends only on the forcing

the characteristic polynomial

solve for $\mathcal{L}x$

$$\mathcal{L}x = \frac{\mathcal{L}f + c(s)}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

characteristic polynomial

the denominator resembles the original differential equation via the relationship

$$\frac{d^j x}{dt^j} \longleftrightarrow s^j$$

the numerator is determined by the forcing term and the initial conditions

polynomials in one variable

step 3 (partial fractions) requires the **linear factors** of the characteristic polynomial

degree

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

coefficients

a **linear factor** $(s - a)$ of $p(s)$ corresponds to a **root** (or a **zero**)

$$p(a) = 0 \text{ if, and only if, } p(s) = (s-a)q(s)$$

a polynomial of degree *n* can have no more than *n* roots, counted with multiplicity

fundamental theorem of algebra

a polynomial of degree n with complex coefficients always has n complex roots:

1777-1855

$$p(s) = a_n(s - z_1)(s - z_2)\cdots(s - z_n)$$

where $z_1, \cdots z_n$ **are complex numbers**

this theorem was first properly stated in Gauss's doctoral thesis in 1799: "*Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse*"

the complex numbers are "algebraically closed"

however, for n > 4, there is no algebraic formula for the roots in terms of the coefficients

solving linear ODEs – partial fractions

express the characteristic polynomial in terms of its roots, collecting together **repeated roots**

$$Z(s) = a_n(s - z_1)^{r_1}(s - z_2)^{r_2} \cdots (s - z_k)^{r_k}$$

	factor in characteristic polynomial	term in partial fraction expansion
root z_i appears once only	$(s - z_i)$	$\dfrac{A_i}{s - z_i}$
root z_j appears r_j times	$(s - z_j)^{r_j}$	$\dfrac{A_{j,1}}{(s - z_j)} + \dfrac{A_{j,2}}{(s - z_j)^2} + \cdots + \dfrac{A_{j,r_j}}{(s - z_j)^{r_j}} = \sum_{u=1}^{r_j} \dfrac{A_{j,u}}{(s - z_j)^u}$

solving linear ODEs – repeated roots

step 3 becomes the following partial fraction expansion

$$\mathcal{L}(x)(s) = \frac{c(s)}{Z(s)} = \left(\sum_{u=1}^{r_1} \frac{A_{1,u}}{(s - z_1)^u} \right) + \cdots + \left(\sum_{u=1}^{r_k} \frac{A_{k,u}}{(s - z_k)^u} \right)$$

the solution can then be read off from the TABLE as in step 4

$$\frac{A_{j,1}}{(s - z_j)} \longrightarrow e^{z_j t}$$

$$\frac{A_{j,u}}{(s - z_j)^u} \longrightarrow t^u e^{z_j t}$$

powers of t in the solution arise from repeated roots of the characteristic polynomial

solving linear ODEs – the general solution

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 = 0$$

has solutions which are linear combinations of terms of the form $t^j e^{z_i t}$

where z_i is a root of the characteristic polynomial

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

and j is less than the number of times z_i is repeated as a root

complex conjugate roots conspire to make the overall solution real

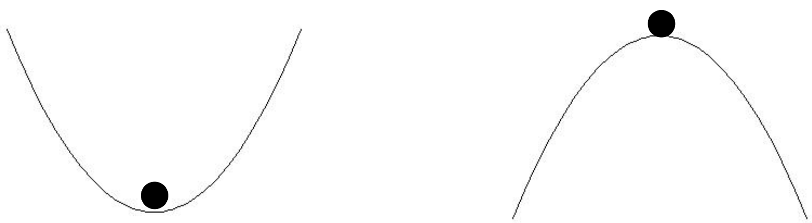
dynamical stability

the stability of a linear ODE is determined by how it behaves in the absence of any forcing

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0$$

when its steady state occurs at x = 0

the system is STABLE if it relaxes back to 0 from any initial condition



stable unstable