dynamic processes in cells (a systems approach to biology)

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lecture 3

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solving linear ODEs – preliminaries

n-th order, homogeneous linear ordinary differential equation (ODE), with constant coefficients,

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0$$

we want to determine the solutions, $x(t)$, of the equation and work out the stability of the steady state at

$$x = 0$$

initial conditions: to uniquely determine a solution, n initial conditions must be specified

$$x(0)\,,\ \frac{dx}{dt}(0)\,,\ \frac{d^2 x}{dt^2}(0)\,, \cdots ,\ \frac{d^{n-1} x}{dt^{n-1}}(0)$$

solving linear ODEs of order 1

the first-order, linear ODE $$\frac{dx}{dt} = ax$$

has the unique solution

initial condition

$$x(t) = e^{at} x(0)$$

where the exponential function $e^t = \exp(t)$ is defined by

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3.2} + \cdots + \frac{t^n}{n!} + \cdots$$

(this definition works generally for complex numbers or matrices *)

* see the lecture notes “Matrix algebra for beginners, Part III”

solving linear ODEs of order 2

the second-order, linear ODE $\frac{d^2x}{dt^2} = -ax \quad a > 0$

has solutions $\cos(\sqrt{a}t)$ and $\sin(\sqrt{a}t)$

$$\cos(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} \qquad \sin(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!}$$

$$e^{it} = \cos(t) + i\sin(t)$$ ← **Euler's formula**


1707-1783

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \qquad \sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

the exponential function over the complex numbers solves ODEs of order 1 and 2

complex numbers

numbers written in the form $a + ib$ with a, b ordinary “real” numbers and $i^2 = -1$

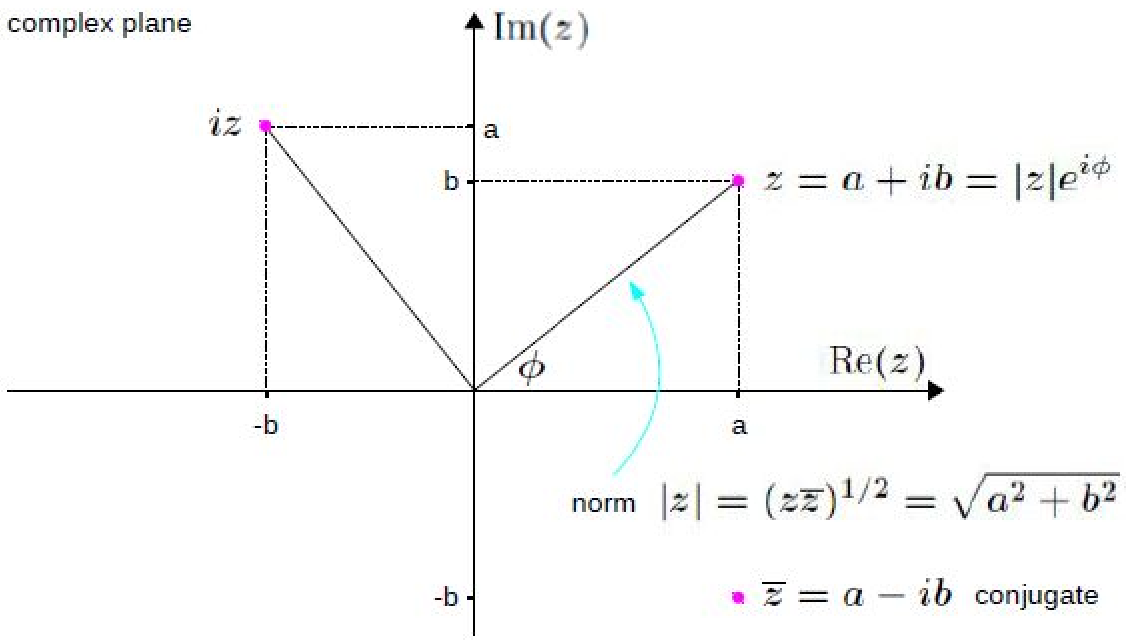


Figure 4.2: The complex plane, or Argand diagram, showing the point $z = a + ib$; its *complex conjugate*, $\overline{z} = a - ib$; its *norm*, $|z|$, or distance from the origin; and the point iz, which is given by counter-clockwise rotation of z through 90 degrees. z can be represented either in terms of its real and imaginary parts, $z = a + ib$, or by using Euler's formula in Eq. 4.13 to write it as $z = |z|e^{i\phi}$, where ϕ is the angle between z and the real axis, so that $a = |z|\cos(\phi)$ and $b = |z|\sin(\phi)$.

solving linear ODEs of order n

THAT IS ALL YOU NEED (WELL, ALMOST*)

the solution of any linear ODE, no matter what its order, is (almost*) a linear combination of complex exponentials

$$\sum_i \lambda_i e^{z_1 t}$$

where z_i are complex numbers determined by the coefficients and λ_i are complex numbers determined by the initial conditions

* you also need powers of t $t^j e^{z_i t}$

the Laplace transform

1749-1827

transforms a function $x(t)$ of t into a function $(\mathcal{L}x)(s)$ of s

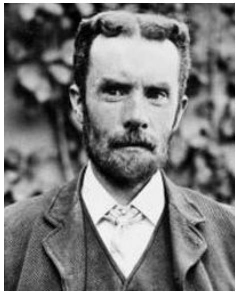
$$(\mathcal{L}x)(s) = \int_0^\infty e^{-st} x(t)dt \quad Re(s) > c$$

the integral is defined ("converges") for **s** sufficiently large, provided $x(t)$ does not increase "too fast"

s may take complex values, in which case the integral is defined for the real part of **s** sufficiently large – so that the Laplace transform is defined in some right half plane of the complex numbers

using it to solve linear ODEs

Heaviside



1850-1925

the Laplace transform converts **differentiation by t** into **multiplication by s**, thereby transforming calculus into algebra

1. **apply the Laplace transform**

2. **express the Laplace transform of the solution as a function of s**

3. **break up this function of s into a linear combination of functions whose Laplace transforms are known (provided in a Table)**

4. **use the Table to write down the solution**

5. **remember Oliver Heavside**

Paul Nahin, **Oliver Heaviside. The Life, Work and Times of an Electrical Genius of the Victorian Age,** Johns Hopkins University Press, 1988; Bush, **Operational Circuit Analysis**, John Wiley, 1929

Vaneevar Bush, **Operational Circuit Analysis**, John Wiley, 1929. Bush wrote that readers would *"frequently turn for inspiration and background to Heaviside's own works, of which this is in some sense an interpretation"*

properties of the Laplace transform

1. **it converts differentiation by t into multiplication by s**

$$\mathcal{L}\left(\frac{df}{dt}\right) = s(\mathcal{L}f) - f(0)$$

initial condition

2. **it converts multiplication by t into differentiation by s**

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}(\mathcal{L}f)(s)$$

properties of the Laplace transform

3. it is linear

$$\mathcal{L}(\lambda_1 x_1(t) + \lambda_2 x_2(t)) = \lambda_1 (\mathcal{L} x_1)(s) + \lambda_2 (\mathcal{L} x_2)(s)$$

4. it is one-to-one (for our purposes)

if $(\mathcal{L} f)(s) = (\mathcal{L} g)(s)$ then $f(t) = g(t)$

$f(t)$	$(\mathcal{L}f)(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{d^n f}{dt^n}$	$s^n(\mathcal{L}f)(s) - \sum_{r=0}^{n-1} s^{n-r-1}\frac{d^r f}{dt^r}(0)$

TABLE of Laplace transforms

↑ initial conditions

solving linear ODEs with the Laplace transform

$$\frac{dx}{dt} + ax = b$$

1. apply the Laplace transform to both sides

$$\mathcal{L}(\frac{dx}{dt} + ax) = \mathcal{L}(b)$$

2. use the properties of $\mathcal{L}$ and the TABLE to simplify and solve for $\mathcal{L}x$

$$s(\mathcal{L}x) + a(\mathcal{L}x) = \frac{b}{s} + x(0)$$

$$(\mathcal{L}x) = \frac{b}{s(s+a)} + \frac{x(0)}{s+a}$$

solving linear ODEs with the Laplace transform

$$(\mathcal{L}x) = \frac{b}{s(s+a)} + \frac{x(0)}{s+a}$$

3. use **partial fractions** to rewrite the RHS in terms of functions in the TABLE

$$\frac{b}{s(s+a)} = \left(\frac{b}{a}\right)\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

4. now use the TABLE again to deduce what the solution must have been

$$x(t) = \frac{b}{a} + (x(0) - \frac{b}{a})e^{-at}$$

solving linear ODEs – partial fractions

3. use partial fractions to rewrite the RHS in terms of functions in the TABLE

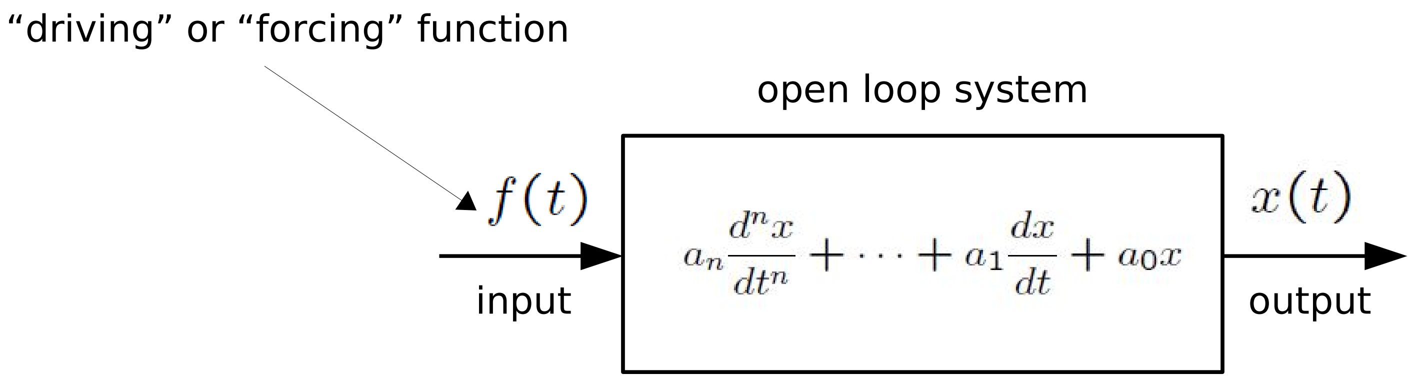
$$\frac{b}{s(s+a)} = \left(\frac{b}{a}\right)\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

this depends on the **linear factors** of the denominator polynomial

$$s^2 + as = s(s+a)$$

which determine where the denominator polynomial becomes 0, at s = 0 and s = -a

solving linear ODEs – the general linear system



$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

solving linear ODEs – apply Laplace transform

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

recall the formula from the TABLE

$$\mathcal{L}\left(\frac{d^j x}{dt^j}\right) = s^j (\mathcal{L}x)(s) - \sum_{u=0}^{j-1} s^{j-u-1} \frac{d^u x}{dt^u}(0)$$

depends only on the initial conditions

apply the Laplace transform to both sides of the equation

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0)(\mathcal{L}x) - c(s) = (\mathcal{L}f)(s)$$

resembles the original differential equation

depends only on the forcing

solving linear ODEs – structure of the solution

solve for $\mathcal{L}x$

$$\mathcal{L}x = \frac{\mathcal{L}f + c(s)}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

characteristic polynomial

the denominator resembles the original differential equation via the relationship

$$\frac{d^j x}{dt^j} \longleftrightarrow s^j$$

the numerator is determined by the forcing term and the initial conditions

step 3 (partial fractions) requires the **linear factors** of the characteristic polynomial

polynomials in one variable

degree

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

coefficients $p(s)$

a **linear factor** $(s - a)$ of $p(s)$ corresponds to a **root** (or a **zero**)

$$p(a) = 0 \quad \text{if, and only if,} \quad p(s) = (s-a)q(s)$$

a polynomial of degree *n* can have no more than *n* roots, counted with multiplicity

the polynomial $x^2 + 1$ has no roots which are real numbers but does have roots which are complex numbers

fundamental theorem of algebra

a polynomial of degree n with complex coefficients always has n complex roots:

$$p(s) = a_n(s - z_1)(s - z_2)\cdots(s - z_n)$$

where $z_1, \cdots z_n$ **are complex numbers**

1777-1855

this theorem was first properly stated in Gauss's doctoral thesis in 1799: "*Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse*"

the complex numbers are "algebraically closed"

if the coefficients of the polynomial are real, then the roots may be complex but they will occur in conjugate pairs z_i and $\overline{z_i}$

$$z = a + ib \qquad \overline{z} = a - ib$$

solving linear ODEs – partial fractions

express the characteristic polynomial in terms of its roots, collecting together **repeated roots**

$$Z(s) = a_n(s - z_1)^{r_1}(s - z_2)^{r_2} \cdots (s - z_k)^{r_k}$$

	factor in characteristic polynomial	term in partial fraction expansion
root z_i appears once only	$(s - z_i)$	$\dfrac{A_i}{s - z_i}$
root z_j appears r_j times	$(s - z_j)^{r_j}$	$\dfrac{A_{j,1}}{(s - z_j)} + \dfrac{A_{j,2}}{(s - z_j)^2} + \cdots + \dfrac{A_{j,r_j}}{(s - z_j)^{r_j}} = \sum_{u=1}^{r_j} \dfrac{A_{j,u}}{(s - z_j)^u}$

solving linear ODEs with no forcing

step 3 becomes the following partial fraction expansion

$$\mathcal{L}(x)(s) = \frac{c(s)}{Z(s)} = \left(\sum_{u=1}^{r_1} \frac{A_{1,u}}{(s - z_1)^u} \right) + \cdots + \left(\sum_{u=1}^{r_k} \frac{A_{k,u}}{(s - z_k)^u} \right)$$

the solution can then be read off from the TABLE as in step 4

$$\frac{A_{j,1}}{(s - z_j)} \longrightarrow e^{z_j t}$$

$$\frac{A_{j,u}}{(s - z_j)^u} \longrightarrow t^u e^{z_j t}$$

powers of t in the solution arise from repeated roots of the characteristic polynomial

in summary, when there is no forcing,

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 = 0$$

has solutions which are linear combinations of terms of the form $t^j e^{z_i t}$

where z_i is a root of the characteristic polynomial

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

and j is less than the number of times z_i is repeated as a root

complex conjugate roots conspire to make the overall solution real

solving linear ODEs with forcing

the Laplace transform of the solution is given by

$$\mathcal{L}(x)(s) = \frac{\mathcal{L}(f) + c(s)}{Z(s)}$$

the forcing term $\mathcal{L}(f)$ may introduce new roots into the denominator

$$\frac{dx}{dt} + ax = b \qquad (\mathcal{L}x) = \frac{b}{s(s+a)} + \frac{x(0)}{s+a}$$

we will see this again in the next lecture with the Bode plot

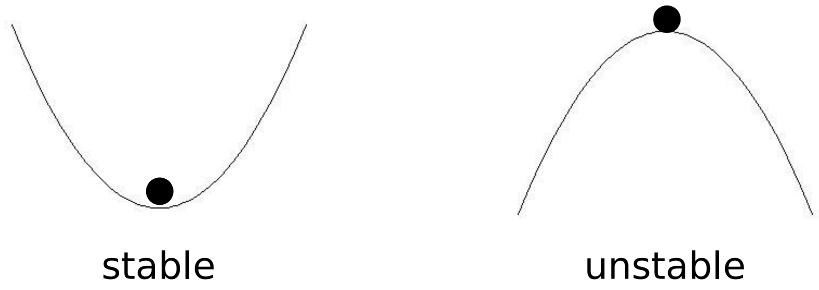
stability, or "preventing wide oscillations"

the stability of a linear ODE is determined by how it behaves in the absence of any forcing

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0$$

when its steady state occurs at x = 0

the system is STABLE if it relaxes back to 0 from any initial condition



stability and the roots of Z(s)

the solution with any initial condition is a linear combination of terms

$$t^j e^{z_u t}$$

$$z_u = a_u + i b_u$$

real part

imaginary part

$$t^j e^{z_u t} = t^j e^{a_u t} e^{i b_u t}$$

Euler's formula – sine or cosine

increasing power of t (if z_u is repeated)

increasing or decaying exponential depending on whether a_u is positive or negative

a decaying exponential of rate *a* > *0*, no matter how small, will always overwhelm a power of *t*, no matter how large,

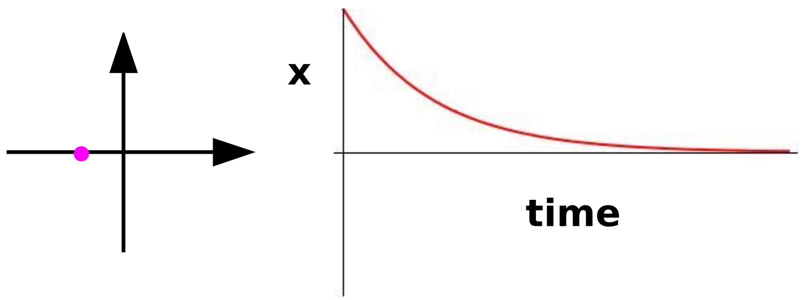
$$t^k e^{-at} \longrightarrow 0$$

hence, the solution with any initial condition will relax to 0 if all the roots of the characteristic equation have negative real parts

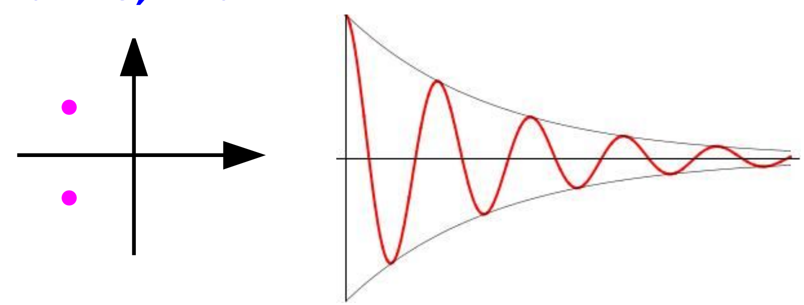
stability depends on the real part of the root

$$e^{(a+ib)t} = e^{at} e^{ibt}$$

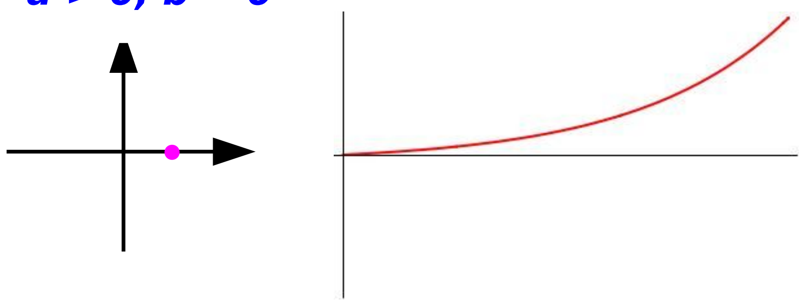
$a < 0, b = 0$



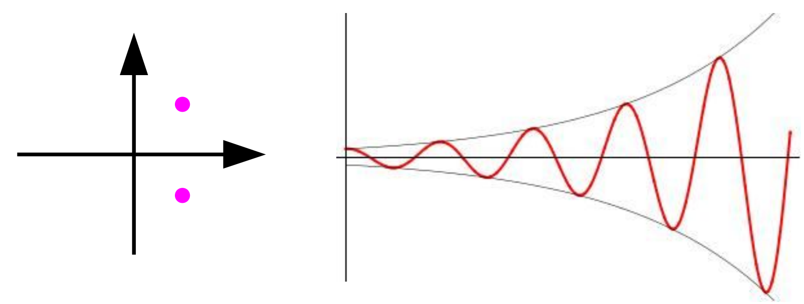
$a < 0, \pm b$



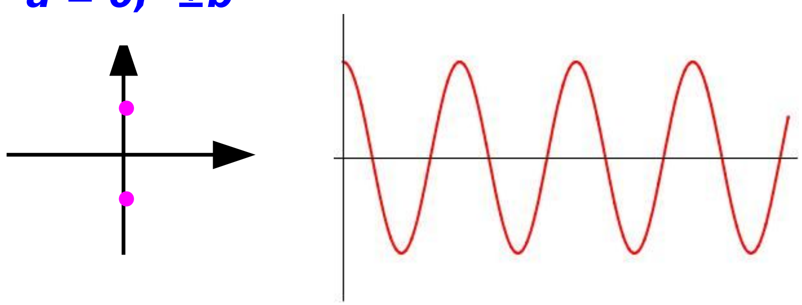
$a > 0, b = 0$



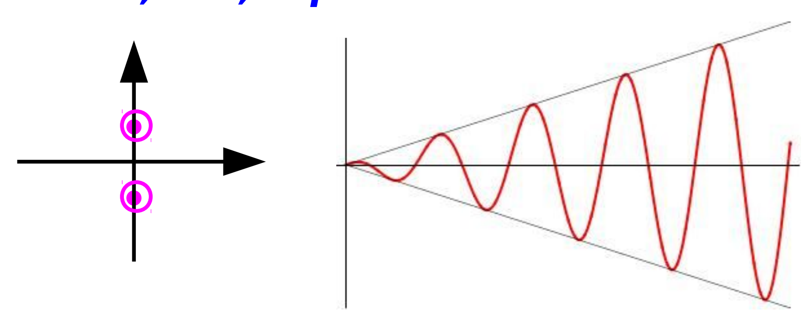
$a > 0, \pm b$



$a = 0, \pm b$



$a = 0, \pm b$, repeated twice



stability of a linear ODE

It will be seen that the motion of a machine with its governor consists in general of a uniform motion, combined with a disturbance which may be expressed as the sum of several component motions. These components may be of four different kinds :-

(1) The disturbance may continually increase.
(2) It may continually diminish.
(3) It may be an oscillation of continually increasing amplitude.
(4) It may be an oscillation of continually decreasing amplitude.

The first and third cases are evidently inconsistent with the stability of the motion; and the second and fourth alone are admissible in a good governor. This condition is mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots, of a certain equation shall be negative.

1831-1879

a linear ODE, with no forcing term, is stable if the roots of its characteristic equation all have negative real parts

J C Mawell, *"On governors"*, Proc Roy Soc, **16**:270-83, 1868.

second-order linear ODEs

consider a system which can be normalised as follows

positive

coefficient of x is +1

$$\left(\frac{1}{\omega^2}\right)\frac{d^2x}{dt^2} + \left(\frac{2\delta}{\omega}\right)\frac{dx}{dt} + x = 0$$

$\omega > 0$	fundamental frequency	$(\text{time})^{-1}$
δ	damping ratio	dimensionless

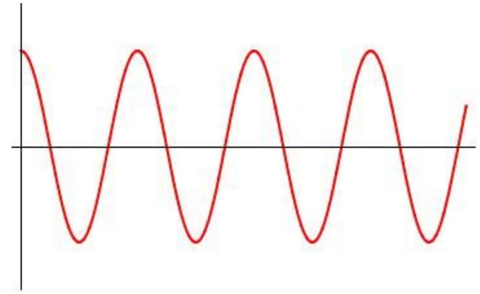
with these choices the characteristic polynomial has the following two roots

$$s = \omega(-\delta \pm \sqrt{\delta^2 - 1})$$

the system is stable provided that $\delta > 0$

$\delta = 0$

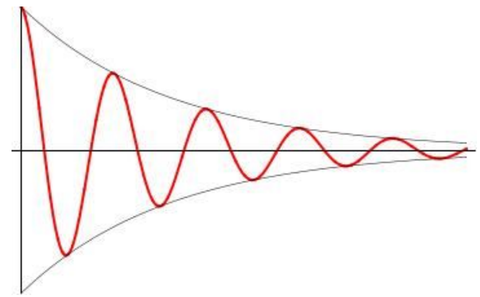
undamped



$\cos(\omega t)$

$0 < \delta < 1$

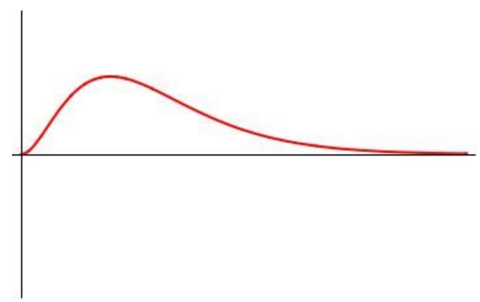
underdamped



$e^{-\omega\delta t}\cos(\omega\sqrt{1-\delta^2}\,t)$

$\delta = 1$

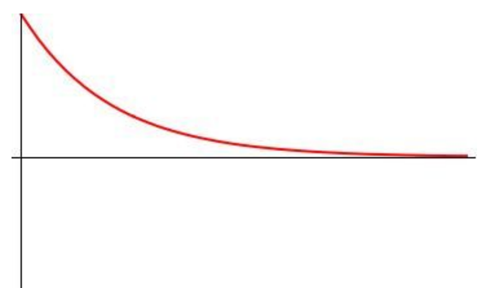
critically damped



$te^{-\omega\delta t}$

$1 < \delta$

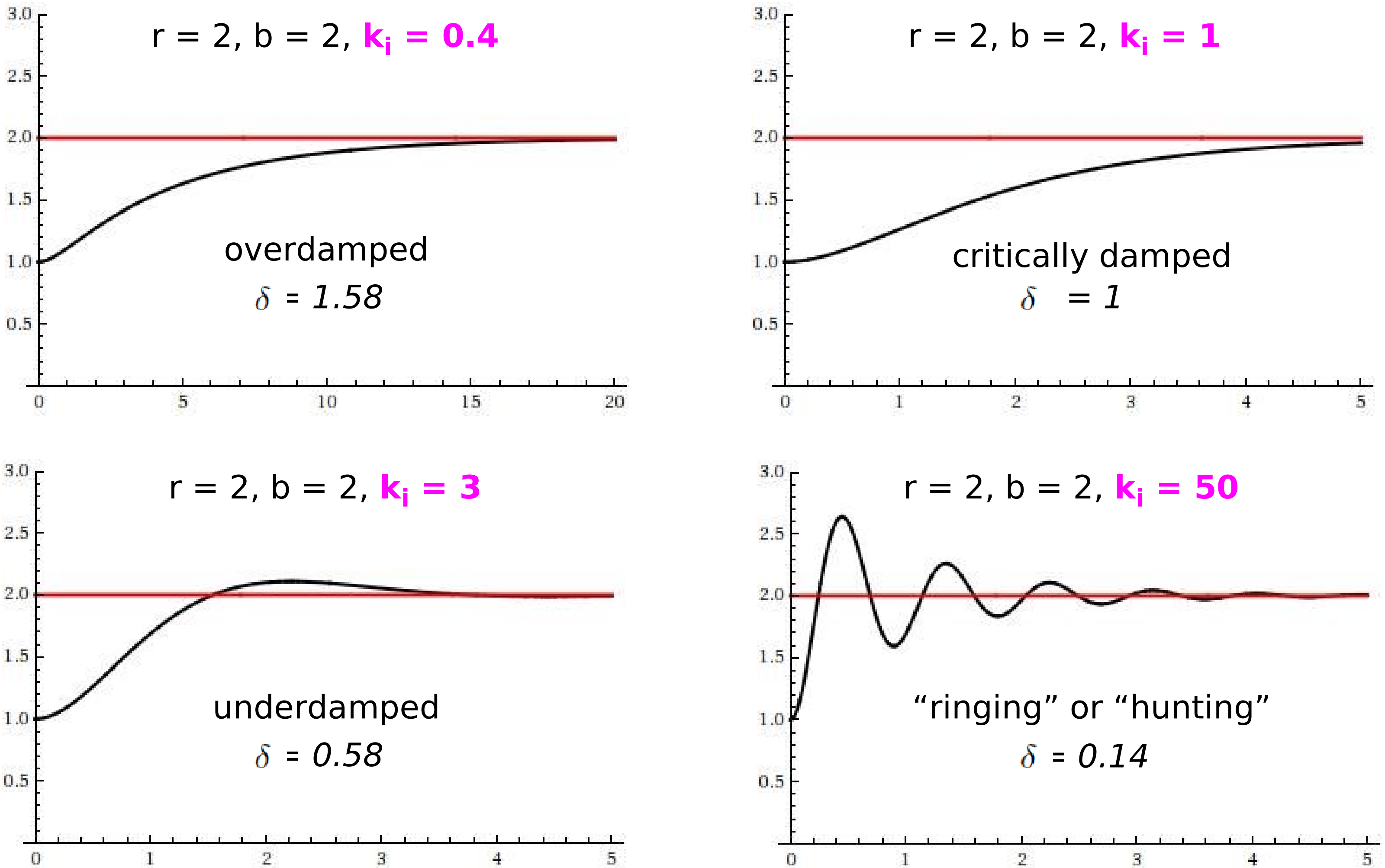
overdamped



$e^{\omega(-\delta+\sqrt{\delta^2-1})t}$

integral controllers

$$\left(\frac{1}{k_i}\right)\frac{d^2x}{dt^2} + \left(\frac{b}{k_i}\right)\frac{dx}{dt} + x = r \qquad \omega = \sqrt{k_i} \quad \delta = \frac{b}{2\sqrt{k_i}}$$



proportional integral controllers

$$\left(\frac{1}{k_i}\right)\frac{d^2x}{dt^2} + \left(\frac{b+k_p}{k_i}\right)\frac{dx}{dt} + x = r \qquad \omega = \sqrt{k_i} \quad \delta = \frac{b+k_p}{2\sqrt{k_i}}$$